Secretary Problem.

Note Title 2/19/2013

- Sequence of secretaries., s_i for $i = 1..n$.
  $\exists$ a preference order, $\sigma : [n] \to \{s_1, \dots s_n\}$
- When s_i arrives, see if $\sigma(s_i) \leq \sigma(s_j)\ \forall\ j = 1..i-1$
  "best so far" (BSF)
- decide "hire" or "no hire" irreversibly.

- Assume s_i's arrive in a random order.
  $\equiv$ assuming σ is a random permutation.
- Maximize probability of hiring best secretary.

Algo:- parameter r.
For $i = 1..r$
- never hire s_i

For $i = r+1..n$
- hire i if i is best so far.

$$ALG_r = \Pr[\text{hire } \sigma_1] = \sum_{i=1}^{n} \Pr[\text{hire } s_i \mid \sigma_1 = s_i] \Pr[\sigma_1 = s_i]$$

$\Pr[\sigma_1 = s_i] = \frac{1}{n}$. If $i \leq r$, $\Pr[\text{hire } s_i] = 0$.

if $i > r$ $\Pr[\text{hire } s_i \mid s_i = \sigma(1)] = \Pr[\text{hire } s_i \mid s_i \text{ is BSF}]$

$= \Pr[\text{not hire } s_1, \dots s_{i-1}]$

$= \Pr[\text{not hire } s_{r+1}, \dots s_{i-1}]$

$= \Pr[r+1, \dots i-1 \text{ are not BSF}]$

$= \Pr[\text{best in } 1..i-1 \text{ lies in } 1..r]$

$= r/(i-1)$.

$$\cancel{A}LG_r = \frac{r}{n} \sum_{i=r}^{n} \frac{1}{i-1}$$

$$= \frac{r-1}{n} \left[H_{n-1} - H_{r-1} \right]$$

$$\approx \frac{r}{n} \log_e \frac{n}{r}$$

$$\frac{d\,ALG}{dr} = \frac{\cancel{r}}{\cancel{n}} \cdot \frac{-1}{\cancel{r}} + \frac{1}{\cancel{n}} \log_e \frac{n}{r} = 0$$

$$\frac{n}{r} = e$$

$$\therefore ALG = \frac{1}{e}.$$

Is this optimal?

Consider any possibly randomized mechanism.
Let $p_i = \Pr[s_i \text{ is hired}]$ where the probability is over the randomness in the mechanism and the random permutation.
w.l.o.g Assume mechanism hires s_i only if s_i is "best-so-far".

(1) —— $p_i = \Pr[\text{hire } s_i \mid s_i \text{ is best so far}] \cdot \Pr[s_i \text{ is BSF}]$.

$$\Pr[s_i \text{ is BSF}] = 1/i.$$

$$\Pr[\text{hire } s_i \mid s_i \text{ is BSF}] \leq \Pr[\text{not hire } s_1, \dots, s_{i-1}]$$
$$= 1 - (p_1 + p_2 + \dots + p_{i-1})$$

$\therefore \ i p_i \leq 1 - (p_1 + p_2 + \cdots + p_{i-1})$

$E[ALG] = \mathbb{P}r[\text{hire } \sigma(1)]$

$= \sum_{i=1}^{n} \mathbb{P}r[\sigma(1) = S_i] \cdot \mathbb{P}r[\text{hire } S_i \mid \sigma(1) = S_i]$

$\mathbb{P}r[\sigma(1) = S_i] = 1/n$

$\mathbb{P}r[\text{hire } S_i \mid \sigma(1) = S_i] = \mathbb{P}r[\text{hire } S_i \mid S_i \text{ is BSF}]$

$\because$ Algo cannot distinguish between

$\sigma(1) = S_i$ & S_i is BSF.

From (1), $\mathbb{P}r[\text{hire } S_i \mid S_i \text{ is BSF}] = i p_i$

$\therefore \ E[ALG] = \sum_{i=1}^{n} \frac{1}{n} i p_i$

$\therefore \ E[ALG] \leq \max \ \frac{1}{n} \sum_{i=1}^{n} i p_i$ s.t.

x_i — $\forall i = 2..n \quad i p_i \leq 1 - (p_1 + \cdots + p_{i-1})$

$\forall i = 1..n \quad p_i \geq 0.$

Dual : $\min \ \sum_i x_i$ s.t.

$\forall i = 1..n \quad i x_i + x_{i+1} + \cdots + x_n \geq \frac{i}{n},$

$x_i \geq 0.$

Dual solution :-

$n x_n \geq \frac{n}{n} \quad \therefore$ set $x_n = \frac{1}{n}$

$(n-1) x_{n-1} + x_n \geq \frac{n-1}{n} \quad \therefore$ set $(n-1) x_{n-1} + \frac{1}{n} = \frac{n-1}{n}$

$\Rightarrow x_{n-1} = \frac{1}{n} - \frac{1}{n(n-1)} = \frac{1}{n}\left[1 - \frac{1}{n-1}\right] = \frac{n-2}{n(n-1)}$

$$(n-2)x_{n-2} + \frac{n-2}{n \cdot n-1} + \frac{1}{n} = \frac{n-2}{n}$$

$$\therefore x_{n-2} = \frac{1}{n} - \frac{1}{n \cdot n-1} - \frac{1}{n \cdot n-2}$$

$$= \frac{1}{n}\left[1 - \frac{1}{n-1} - \frac{1}{n-2}\right]$$

$$i x_i + x_{i+1} + x_{i+2} + \dots + x_n = \frac{i}{n}$$

$$- \left[(i+1) x_{i+1} + x_{i+2} + \dots + x_n = \frac{i+1}{n}\right]$$

$$\Rightarrow i x_i - i x_{i+1} = -\frac{1}{n}$$

$$\Rightarrow x_i = x_{i+1} - \frac{1}{ni}$$

$$= \frac{1}{n}\Big[1 - \underbrace{\frac{1}{n-1} - \frac{1}{n-2} \dots - \frac{1}{i+1} - \frac{1}{i}}_{\leq 1}\Big]$$

$$\frac{1}{n-1} + \frac{1}{n-2} + \dots + \frac{1}{r} = H_{n-1} - H_{r-1} \approx \log_e \frac{n}{r} = 1$$

when $r = n/e$.

$\therefore x_i = 0 \quad \forall\, i \leq n/e.$

Dual objective $= \sum_i x_i = \sum_{i \geq r} x_i$

$$r x_r + x_{r+1} + \dots + x_n \simeq \frac{r}{n}$$

$$x_r = 0. \quad \therefore x_{r+1} + \dots + x_n \simeq \frac{r}{n} = \frac{1}{e}$$

$\square$

Also the p_i's corresponding to the algo is

$p_i = 0 \;\; \forall\, i = 1 \dots \lfloor n/e \rfloor \quad p_i = \frac{r}{i(i-1)} \;\; \forall\, i = \lfloor n/e \rfloor + 1, \dots, n$

Incentive Compatibility: No incentive for a secretary to move up/down the order.

$\Rightarrow$ $\Pr[\text{hire } s_i]$ is the same for all $i = p$

No longer true that only hire S_i if it is BSF.

$\therefore$ Let $f_i = \Pr[\text{hire } S_i \mid S_i \text{ is BSF}] = \Pr\left[\frac{\text{hire } S_i}{S_i = \text{BSF}}\right]$

$p = \Pr[\text{hire } s_i] \geq f_i \cdot \frac{1}{i}$.

$f_i \leq 1 - (i-1)p$.

$p \leq \frac{1}{n}$.

$\max \sum_{i=1}^{n} \frac{1}{n} f_i$

Given p, set $f_i = \min\{ip, 1-(i-1)p\}$

$ip = 1-(i-1)p \Rightarrow 2ip = 1-p$

$\therefore i = \frac{1-p}{2p} = \frac{1}{2p} - \frac{1}{2}$

$\therefore f_i = \begin{cases} ip & \text{if } i \leq \frac{1}{2p} - \frac{1}{2} \\ 1-(i-1)p & \text{o.w.} \end{cases}$

Mechanism:-

for $i \leq \frac{1}{2p}$, if you reach s_i, & s_i is BSF, Then hire S_i w.p.

$f_i / [1 - \Pr(\text{reach } s_i)] = f_i / 1-(i-1)p$.

$$\frac{f_i}{1-(i-1)p} = \frac{ip}{1-(i-1)p} = \frac{i}{\frac{1}{p} - i + 1}$$

for $i > \frac{1}{2p}$, hire w/ some probability even if not BSF.